Fakultät für Mathematik
Institut für Mathematische Optimierung
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Examination in Mathematics I

(24.07.2002)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- printed script “Mathematics for Students of Economics and Management”

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	6	5	8	14	8	9	50

Problems:

1. Given is the complex number $z = 2 - 2i$.
Find $z_1 = z^{10}$ and $z_2 = \frac{z}{\bar{z}}$. Give the results in cartesian coordinates.

2. Given is the sequence $\{a_n\}$ with $a_n = 6\frac{2^n}{3^{n-1}}$.
   (a) Find $\lim_{n\to\infty} a_n$.
   (b) Check whether the series $\sum_{n=1}^{\infty} a_n$ converges. Use quotient criterion.

3. Let $P(x) = x^4 - 5x^3 + 3x^2 + 9x$.
   (a) Determine all real zeros of $P(x)$.
   (b) The function $f(x)$ is defined as $f(x) = \frac{x^2 - 1}{P(x)}$. What is the domain of $f(x)$?
   (c) Let $x_0 = -1$. Does exist $f(x_0)$? Find $\lim_{x\to x_0} f(x)$.

4. Consider the function $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}$ with
$$f(x) = (x - 1)[\ln(x - 1)]^2.$$
   (a) Find domain of f, extreme points and inflection points.
   (b) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to 1+0} f(x)$.

5. A firm has the total-cost function
$$C(x) = \frac{1}{3}x^3 - 7x^2 + 111x + 50$$

where x is the output (the number of units produced). The demand x which is equal to the output depends on price p only:

$$x = f(p) = 100 - p.$$

(a) Write out the total-revenue function $R = x \cdot p$ in terms of x and formulate the total-profit function $G(x) = R(x) - C(x)$ in terms of the output x.

(b) Find the profit-maximizing level of output x_0 and the maximum profit $G_{max}(x_0)$.

(c) Formulate the total-profit function $G(p) = x \cdot p - C(x)$ in terms of the price variable p.

(d) Show that the price $p_0 = 100 - x_0$ maximizes $G(p)$.

6. (a) Find

$$\int \frac{21e^{2x}}{\sqrt[4]{e^x + 1}}\, dx$$

(b) Evaluate

$$\int_1^e \frac{\ln t}{t^3}\, dt$$

Solutions Mathematics I (24.07.2002):

1. $z_1 = -2^{15}i; \quad z_2 = -i$

2. (a) $\lim\limits_{n\to\infty} a_n = 0$

   (b) $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{2}{3} < 1$. Series converges.

3. (a) $x_1 = 0, \quad x_2 = -1, \quad x_3 = x_4 = 3$

   (b) $D_f = \{x \in \mathbb{R} \,|\, x \neq 0 \wedge x \neq -1 \wedge x \neq 3\}$

   (c) $f(-1)$ does not exist. $\lim\limits_{x\to-1} f(x) = \frac{1}{8}$

4. (a) $D_f = \{x \in \mathbb{R} \,|\, x > 1\}$
   local maximum at $(e^{-2}+1, 4e^{-2})$, local minimum at $(2,0)$
   inflection point at $(e^{-1}+1, e^{-1})$

   (b) $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to1+0} f(x) = 0$

5. (a) $G(x) = x(100-x) - \left(\frac{1}{3}x^3 - 7x^2 + 111x + 50\right)$

   (b) $x_0 = 11; \quad G_{max}(11) = 111.33$

   (c) $G(p) = (100-p)p - \left(\frac{1}{3}(100-p)^3 - 7(100-p)^2 + 111(100-p) + 50\right)$

   (d) $G'(p_0) = G'(89) = 0$ and $G''(89) < 0$.

6. (a) $\displaystyle\int \frac{21e^{2x}}{\sqrt[4]{e^x+1}}\,dx = \sqrt[4]{(e^x+1)^3}(12e^x - 16) + C$

   (b) $\displaystyle\int_1^e \frac{\ln t}{t^3}\,dt = -\frac{3}{4e^2} + \frac{1}{4} = 0.1485$