Original

Final Exam: 1798 Mathematical Economics, Summer 2006

No aids permitted except for (1) language dictionaries without any marking, (2) non-programmable pocket calculators without communicating and/or data processing functions, and (3) the hand out of the collection of formulas.

There are five problems on this exam. Solve all of them.

1. (30 points) Consider the following nonlinear programming problem where $x_1, x_2 \in [0, \infty)$:

$$\max_{x_1,x_2} \quad (1+x_1)\cdot(2+x_2)$$

subject to

$$x_2 + 4x_1 \leq 20$$
$$2 + 0.5x_1 \geq x_2$$
$$x_1, x_2 \geq 0$$

   (a) (8) Provide a graphical representation of (i) the constraint set that contains all feasible solutions and (ii) the contour curves of the objective function. Solve the maximization problem geometrically.

   (b) (3) Does the Weierstrass theorem ensure that a solution to this problem exist?

   (c) (3) Does there exist a solution (x_1^*, x_2^*) of the unconstrained problem where the given constraints are ignored?

   (d) (4) Are the Kuhn-Tucker-conditions necessary and/or sufficient for a local/global optimum?

   (e) (8) Set up the Lagrangian and the Kuhn-Tucker-conditions assuming that $x_1, x_2 > 0$. Find all candidates that satisfy the Kuhn-Tucker-conditions.

   (f) (4) What is the optimum? Verify that the proposed optimum candidate satisfies all Kuhn-Tucker conditions.

2. (30 points) Consider a principal-agent problem under certainty where the principal offers the agent a contract (w, e) that specifies the agent's wage $w \geq 0$ and effort $e \in [0, \bar{e}]$ where $\bar{e} > 0$ is the highest level of effort that the agent can exert. If the agent accepts the contract, he produces a service that the principal values at $v(e)$ where $v'(e) > 0$, $v''(e) < 0$, and $v(0) = 0$. If the agent exerts effort, he suffers disutility that is given by $c(e)$ where $c'(e) > 0$, $c''(e) > 0$, and $c(0) = 0$. Furthermore $v'(0) > c'(0)$. The principal maximizes his profit $\pi(w, e) = v(e) - w$ and the agent accepts the principal's contract if his cost does not exceed his benefit. It follows that the maximization problem of the principal is given by:

$$\max_{w,e} \qquad \pi(w,e) = v(e) - w$$

subject to:

$$\begin{aligned} w &\geq c(e) \\ e &\leq \bar{e} \\ w, e &\geq 0 \end{aligned}$$

   (a) (2) Briefly indicate why the principal chooses a contract such that effort and wage are strictly positive, i.e. $w > 0$ and $e > 0$.

   (b) (10) Obtain a geometric solution of the problem in e-w-space (e on horizontal axis, w on vertical axis) using the implicit function theorem to characterize the shape (slope and convexity/concavity) of the contour curves of the objective function. Distinguish between two qualitatively different solutions, one where both constraints are active and another one where the number of active constraints is smaller than 2. Provide a geometric sketch for each type of solution.

(c) (3) Set up the Lagrangian and give the Kuhn-Tucker conditions assuming that $w, e > 0$.

(d) (15) Solve the problem analytically:

   (i) Show that the agent's participation constraint, i.e. $w \geq c(e)$, is always active.

   (ii) Characterize both qualitatively different solutions.

   (iii) Briefly interpret the identifed optimality condition for each case.

3. (15 points) Suppose that two interdependent markets jointly evolve over time. However, the steady state of the underlying dynamic system is characterized by the following system of equations:

$$\begin{aligned} p_1 + 2p_2 - 150 + \delta &= 0. \\ \delta - 80 + 2p_2 &= 0. \end{aligned}$$

   (a) (2) Write the system in matrix notation $Ap = b$ where A is the matrix of coefficients, p is the vector of prices and b is 2×1.

   (b) (8) Does there exist a solution to the linear system? If yes, solve it using matrix algebra, i.e. find $p = A^{-1}b$.

   (c) (5) Use the implicit function theorem to determine the effects of infinitesimally small variations of δ on steady state prices.

4. (15 points) Consider the following nonlinear differential equation:

$$\dot{y} - 6y^{0.5} + 2y = 0$$

   (a) (3) Find all steady states of the given differential equation.

   (b) (4) Geometrically determine the stability property of each steady state.

   (c) (3) Verify that linearizing the differential equation around the stable steady state leads to

$$\dot{y} + y - 9 = 0.$$

   (d) (5) Find the definite solution for the linearized differential equation given the initial condition $y(0) = 1$. Sketch the behavior of $y(t)$ over the long-run, i.e. as time tends to infinity.

5. (10 points) Consider the difference equation

$$x_t - a \cdot x_{t-1} + 1 = 0.$$

   (a) (4) Determine if one or more steady states exist if (i) $a = -1$ and (ii) $a = 1$. For each steady state, determine its stability properties.

   (b) (6) Find the definite solution for the difference equation given the initial condition $x(0) = 1$ and $a = 0.5$. Sketch the behavior of x_t over the long-run, i.e. as time tends to infinity.