Fakultät für Mathematik
Institut für Mathematische Optimierung
Prof. Dr. F. Werner

Examination in Mathematics II

(17.07.2001)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- printed script "Mathematics for Students of Economics and Management"

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	7	11	9	8	8	7	50

Problems:

1. Given are the three vectors

$$x^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \; x^{(2)} = \begin{pmatrix} 1 \\ a \\ 1 \end{pmatrix}, \; x^{(3)} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

   (a) For which values of a are $x^{(1)}$ and $x^{(2)}$ orthogonal?

   (b) For which real numbers a do the vectors constitute a basis in $\mathbb{R}^3$?

   (c) Let $a = 1$. Then the vector $x^{(4)} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$ can be expressed as a linear combination of the vectors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$. Find the linear combination and decide which three of the four vectors $x^{(i)}, i = 1, 2, 3, 4$, constitute a basis in $\mathbb{R}^3$.

2. Given is a matrix $A = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 0 & a \\ 2 & a & 2a \end{pmatrix}, a \in \mathbb{R}$.

   (a) For what values of a does the system of linear equations $Ax = 0$ have a unique solution, several solutions and no solution, respectively? Find the solution(s) for $a = 0$.

   (b) Let $a = 0$.
   Find the eigenvalues of A and those eigenvectors which are associated with the eigenvalue $\lambda_1 = 0$.

3. Given is the following system of linear inequalities:

$$\begin{array}{rcrcr} & & x_2 & \leq & 32 \\ 2x_1 & + & x_2 & \leq & 30 \\ -x_1 & + & x_2 & \geq & -6 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

   (a) Graph the region of all feasible solutions.

   (b) Find all basic feasible solutions by pivoting.

   (c) Describe a solution of the system as a convex combination.

4. Find all stationary points of the function $f(x, y) = (x^2 - y)e^{x+y}$ and check whether they are extreme points.

5. A firm sells x_1 and x_2 units of the two goods G_1 and G_2, respectively. The profit obtained is given by the function

$$P = P(x_1, x_2) = -x_1^2 - 2x_1x_2 - x_2^2 + 10x_1 + 20x_2.$$

Due to technical limitations, suppose there is a constraint

$$ax_1 + bx_2 = c$$

for the production, where a, b, c are real numbers with $a \neq b$.
Find all solutions (x_1, x_2) maximizing P.

6. Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + y = e^t$$

so that $y(0) = 1$ and $y^{'}(0) = 5.5$.

Examination in Mathematics II - Solutions

(17.07.2001)

1. (a) $a = -\frac{1}{2}$

   (b) basis for all $a \in \mathbb{R} \wedge a \neq \frac{4}{3}$

   (c) $\mathbf{x}^{(4)} = 1\mathbf{x}^{(1)} - 2\mathbf{x}^{(2)} + 0\mathbf{x}^{(3)}$
   basises in $\mathbb{R}^3$: $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}\}, \{\mathbf{x}^{(4)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}\}, \{\mathbf{x}^{(1)}, \mathbf{x}^{(4)}, \mathbf{x}^{(3)}\}$

2. (a) The system is always consistent since it is homogeneous.
   For $a = 0 \vee a = -2$ the system has 1 degree of freedom, there exist infinitely many solutions.
   A unique solution exists for $a \neq 0 \wedge a \neq -2$.
   Solution for $a = 0$: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}; t \in \mathbb{R}$

   (b) Eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = \frac{3}{2} + \frac{\sqrt{29}}{2}, \quad \lambda_3 = \frac{3}{2} - \frac{\sqrt{29}}{2}$

   eigenvectors associated with $\lambda_1 : t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}; t \in \mathbb{R}; t \neq 0$

3. (a)

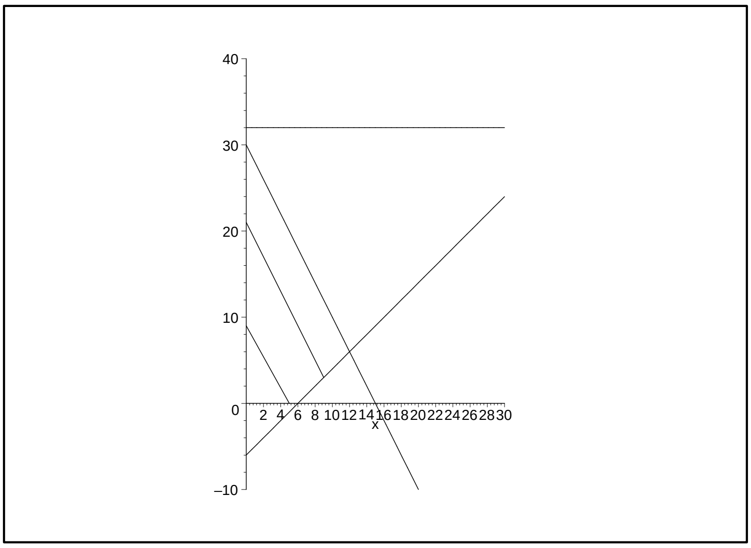

Abbildung 1:

   (b) See (c) for the results

(c) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 12 \\ 6 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 30 \end{pmatrix}$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1; \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$

4. The stationary point $P(-\frac{1}{2}, -\frac{3}{4})$ is no extreme point.

5. The stationary point with

$$x_1 = \frac{-c}{b-a} - \frac{5a^2}{(b-a)^2} + 5, \quad x_2 = \frac{c-5a}{b-a} + \frac{5a^2}{(b-a)^2}$$

is a maximum. The bordered Hessian is $\begin{pmatrix} 0 & a & b \\ a & -2 & -2 \\ b & -2 & -2 \end{pmatrix}$.

6. $y = \frac{1}{2}\cos t + 5\sin t + \frac{1}{2}e^t$