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Examination in Mathematics II

(22.07.2002)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- printed script "Mathematics for Students of Economics and Management"

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	10	11	8	7	8	6	50

Problems:

1. Given is the matrix equation $B + X \cdot A = I$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & \alpha \\ 1 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 2 & 0 \\ -5 & 3 & \alpha \\ 1 & 1 & -5 \end{pmatrix},$$

   I is the identity matrix and $\alpha \in \mathbb{R}$.

   (a) Solve the given equation for X.

   (b) For which values of α does the solution X exist?

   (c) Calculate X.

2. Given is the system of linear equations:

$$\begin{array}{rcrcrcrcl} \lambda x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ x_1 & + & \lambda x_2 & + & x_3 & + & x_4 & = & 1 \\ x_1 & + & x_2 & + & \lambda x_3 & + & x_4 & = & 1 \\ x_1 & + & x_2 & + & x_3 & + & \lambda x_4 & = & 1 \end{array}$$

   (a) Check the consistence of the system as a function of the parameter λ.

   (b) Give the general solution for $\lambda = 1$.

   (c) Find the solution for $\lambda = 0$.

3. Given is the following system of linear inequalities:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & 5x_3 & \leq & 12 \\ 2x_1 & + & x_2 & - & 4x_3 & \leq & 8 \\ 9x_1 & + & 4x_2 & - & 18x_3 & \leq & 45 \\ & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

   (a) Find a first basic feasible solution with x_1, x_2 and x_3 as nonbasic variables.

   (b) Determine a second basic feasible solution with x_1 as basic variable by pivoting.

(c) Assume the constraints given in (a) are part of a linear programming problem with the objective function

$$z = 15x_1 + 7x_2 - 32x_3 \to \max!$$

Check whether your second basic feasible solution calculated in (b) is an optimal solution for the linear programming problem.

4. Given is the function $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^3$, with

$$f(x) = x_1 x_2 (e^{x_2})^2 + \ln(x_3\sqrt{x_2 + 1}) + \frac{x_1}{x_3} - x_1 - \frac{3}{2}x_2.$$

(a) Find the domain D_f.

(b) Determine $\text{grad} f(x^{(0)})$ for $x^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

5. Check whether the function f with

$$f(x, y, z) = x^2 + 3y^2 + 2z^2$$

subject to

$$4x + 12y = 120 \quad \text{and} \quad 6y + 12z = 120$$

has a local minimum at $x = 6, y = 8, z = 6$.
Use Lagrange Multiplier Method.

6. The supply and demand functions for a good are

$$q^S(p) = \frac{2}{3}p - 4; \quad q^D(p) = 20 - 2p.$$

Assume that the price $p = p(t)$ adjusts over time according to the equation

$$\frac{dp}{dt} = p' = [x(p)]^3,$$

where $x(p) = q^D(p) - q^S(p)$ is the excess demand. The initial price is $p(0) = 6$.
Find a formula for $p(t)$. The result can be given implicitly as $F(t, p) = 0$.

Solutions Mathematics II (22.07.2002):

1. (a) $X = (I - B)A^{-1}$
   (b) $\alpha \neq 0$
   (c) $A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2\alpha} & \frac{1}{\alpha} & \frac{1}{2\alpha} \end{pmatrix}$

2. (a) The system is consistent for $\lambda \neq -3$. For $\lambda = 1$ the set of solutions has the dimension 3, otherwise there is a unique solution.
   (b) $x_1 = 1 - x_2 - x_3 - x_4$ with $x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 \in \mathbb{R}$
   (c) $x_1 = x_2 = x_3 = x_4 = \frac{1}{3}$

3. (a) $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 12, x_5 = 8, x_6 = 45$
   (b) $x_1 = 4, x_2 = 0, x_3 = 0, x_4 = 8, x_5 = 0, x_6 = 9$
   (c) The solution is optimal for the linear programming problem.

4. (a) $D_f = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \, x_1 \in \mathbb{R} \wedge x_2 > -1 \wedge x_3 > 0\}$
   (b) $\text{grad} f(x^{(0)}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5. The function has there a local minimum since $\text{grad} L = \mathbf{0}$ and $|\bar{H}_5| > 0$

6. $\dfrac{3}{16\left(24 - \frac{8}{3}p\right)^2} = t + \dfrac{3}{2^{10}}$