Fakultät für Mathematik
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Examination in Mathematics II

(21.07.2003)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- printed script "Mathematics for Students of Economics and Management"

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	7	9	8	11	8	7	50

Problems:

1. (a) Do the three vectors

$$\mathbf{a}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{a}^{(2)} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \ \text{and} \ \mathbf{a}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$$

constitute a basis in $\mathbb{R}^3$?

(b) Express vector $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$ as a linear combination of the vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, $\mathbf{a}^{(3)}$.

(c) Find all basises for the 3-space which include $\mathbf{b}$ and vectors from above.

2. Given is the following system of linear equations:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 6 \\ & & 2\,x_2 & - & x_3 & = & 1 \\ 2\,x_1 & + & 4\,x_2 & - & 3\,x_3 & = & 1 \\ x_1 & - & x_2 & - & \lambda\,x_3 & = & 8 \end{array}$$

(a) Check the consistence of the system as a function of the real parameter λ.

(b) Let $\lambda = -2$. Can you give a solution?

(c) Find the solution for $\lambda = -3$.

3. Given is the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 2 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

(a) Verify that $\lambda_1 = 1$ is an eigenvalue of $\mathbf{A}$ and that

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

is an eigenvector associated with λ_1.

(b) Find all eigenvalues of $\mathbf{A}$.

4. Given is the following linear programming problem:

$$z = x_1 + 4\,x_2 \to \max!$$

subject to

$$\begin{array}{rcrcl} x_1 & - & 2\,x_2 & \leq & 4 \\ -x_1 & + & 2\,x_2 & \leq & 4 \\ x_1 & + & 2\,x_2 & \leq & 8 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

(a) Solve the problem graphically.

(b) Find the optimal solution by the simplex algorithm (pivoting).

5. Given is the function $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^3$, with

$$f(x,y,z) = x^2 y + 2x^2 - 3y^2 - 18y + \ln z - \frac{1}{3}z.$$

(a) Find all stationary points of f.

(b) Check by means of the Hessian whether the function has a local minimum or maximum at $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$.

6. The elasticity $\varepsilon_f(x)$ of a function $y = f(x)$ is given as

$$\varepsilon_f(x) = 2\,x^2 \left(\ln x + \frac{1}{2} \right).$$

Find $f(x)$ as the general solution of a differential equation and determine a particular solution $f(x)$ which satisfies $f(1) = 1$.