Fakultät für Mathematik
Institut für Mathematische Optimierung
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Examination in Mathematics II

(12.02.2002)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- printed script "Mathematics for Students of Economics and Management"

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	7	7	11	8	9	8	50

Problems:

1. Given are the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}.$$

   (a) Find all vectors $\mathbf{x}$ such that $\mathbf{x} \perp \mathbf{a}$ and $\mathbf{x}^{\mathbf{T}}\mathbf{b} = 10$.

   (b) From the solution of (a) determine all vectors with $|\mathbf{x}| = \sqrt{26}$.

2. Let a matrix A and a vector $\mathbf{b}$ be given as follows:

$$A = \begin{pmatrix} 1 & 2 & 4 & -3 \\ 4 & 5 & -2 & 3 \\ 3 & 5 & 6 & -4 \\ 3 & 8 & 24 & -19 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ \alpha \\ 8 \\ 2 \end{pmatrix}.$$

   (a) Is the system of linear equations $A\mathbf{x} = \mathbf{b}$ consistent for all $\alpha \in \mathbb{R}$?

   (b) Evaluate $\det A$.

   (c) Verify that

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

   is an eigenvector of A corresponding to the eigenvalue $\lambda = 0$.

3. (a) A painter decides to blend two kinds of paint (I and II) available for him, in order to meet the following three requirements:

      i. The paint must possess a viscosity of at least 200 centipoises.

      ii. For a desired level of brilliance, there must be at least 14 g of a chemical ingredient Y in each gallon of the paint.

      iii. For a desired degree of durability, at least 30 g of another chemical Z must be present in each gallon of the paint.

   The specifications of paints I and II are the following:

	Paint I (per gallon)	Paint II (per gallon)
Viscosity(centipoises)	400	100
Y (grams)	20	10
Z (grams)	20	60

i. Formulate a mathematical model for finding the feasible portions of paints I and II (call these x_1 and x_2) in each gallon of the blend.

ii. Solve the problem graphically.

(b) Given is the following system of linear inequalities:

$$\begin{array}{rcl} x_1 + x_2 & \leq & 30 \\ 2x_1 + x_2 & \leq & 40 \\ x_1, x_2 & \geq & 0 \end{array}$$

i. Find all basic feasible solutions by pivoting.

ii. Describe a solution of the system as a convex combination.

4. A small workpiece consists of a circular cylinder with an attached cone. Let r be the radius, h be the height of the cylinder and r be the height of the cone. The volume of the workpiece is given by

$$V = \pi r^2 h + \frac{1}{3}\pi r^3.$$

(a) Assume the radius of $r = 2cm$ varies between $1.9cm$ and $2.1cm$ and the height of $3cm$ varies between $2.9cm$ and $3.1cm$. How does the volume vary? Use total differential.

(b) Find the percentage error of the volume when r and h each may vary by 10%.

5. Find all stationary points of the function $f : \mathbb{R}^3 \to \mathbb{R}$ with

$$f(x, y, z) = xy^2 - x^2 - \ln(3y) + z^2$$

and check whether they are extreme points.

6. Find the solution of the differential equation

$$y'' + y' - 6y = 6x$$

such that $y(0) = -\frac{1}{6}$ and $y'(0) = -2$.

Examination in Mathematics II - Solutions

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1. (a) $\mathbf{x} = \begin{pmatrix} 4+t \\ -2-2t \\ t \end{pmatrix}; \quad t \in \mathbb{R}$

   (b) $\mathbf{x_1} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}; \quad \mathbf{x_2} = \frac{1}{3}\begin{pmatrix} 13 \\ -8 \\ 1 \end{pmatrix}$

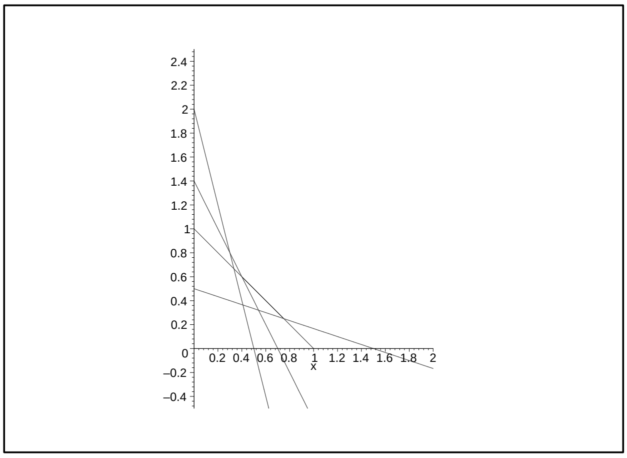
2. (a) No, it is consistent only for $\alpha = 14$.

   (b) $detA = 0$

   (c) It holds $Ax = \lambda x$.

3. (a) i.
$$\begin{array}{rcrcr} 400\,x_1 & + & 100\,x_2 & \geq & 200 \\ 20\,x_1 & + & 10\,x_2 & \geq & 14 \\ 20\,x_1 & + & 60\,x_2 & \geq & 30 \\ x_1 & + & x_2 & = & 1 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

   ii. Feasible solutions are only the points on the straight line $x_1 + x_2 = 1$ between $P_1(0.4; 0.6)$ and $P_2(0.75; 0.25)$; P_1 and P_2 included. (In the figure the resulting lines are drawn.)



   (b) i. $x^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x^{(2)} = \begin{pmatrix} 0 \\ 30 \end{pmatrix}$, $x^{(3)} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$, $x^{(4)} = \begin{pmatrix} 20 \\ 0 \end{pmatrix}$

ii. $\mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 30 \end{pmatrix} + \lambda_3 \begin{pmatrix} 10 \\ 20 \end{pmatrix} + \lambda_4 \begin{pmatrix} 20 \\ 0 \end{pmatrix},$
$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1; \quad \lambda_i \geq 0 \quad \forall i = 1, 2, 3, 4$

4. (a) $dV = 2\pi \; cm^3$
   (b) 30%

5. Stationary point: $P_1(\frac{1}{2}; 1; 0)$, no extreme point,
   for $P_2(\frac{1}{2}; -1; 0)$ function f is not defined.

6. $y_p = -\frac{1}{5}e^{2x} + \frac{1}{5}e^{-3x} - x - \frac{1}{6}$