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Examination:	Mathematical Economics	**Lecture Number:** 1350
Examiner:	Dr. G. Groh	**Wintersemester 2001/2002**
Hint:	75 of the 100 points attainable are regarded as the maximum number one can reach in the time available.	

The following aids can be used: Electronic calculator

Examination questions:

1. (22 points: (a): 8, (b): 6, (c): 8) Consider the following nonlinear programming problem:

$$\begin{aligned} \max_{x_1,x_2} \quad & 25-(x_1-3)^2-(x_2-6)^2 \\ \text{subject to} \quad x_1+x_2 &\leq 5 \\ 3x_1+6x_2 &\geq 12 \\ x_1 &\geq 0, \quad x_2 \geq 0 \end{aligned}$$

   (a) Give a graphical representation of
   - the two constraints
   - the resulting set of feasible solutions and
   - the contour curves of the maximand function

   and try to identify the solution.

   (b) Check, whether *all* conditions are fulfilled which ensure, that the first-order-conditions (i.e. the Kuhn-Tucker-conditions) are *necessary and sufficient* for a global constrained maximum.

   (c) Set up the Kuhn-Tucker-conditions and verify, that the solution obtained in (a) does indeed fulfill them. Don't forget also to determine the values of the Lagrange-multipliers. (Hint: Even if you could not determine the concrete numerical values for x_1^* and x_2^* in (a), the picture nevertheless provides you with enough information to compute these values with the aid of the Kuhn-Tucker-conditions.)

2. (20 points: (a): 7, (b): 8, (c): 5) Consider the following classical optimization problem:

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & -2x_1^2-3x_2^2+3x_1x_2+30x_3 \\ \text{subject to} \quad x_1+x_2+x_3 &= 10 \\ x_1,x_2 &\in \mathbf{R} \end{aligned}$$

   (a) Set up the Lagrange function and compute the solution via the corresponding first-order-conditions.

   (b) Check with the aid of the bordered Hessian matrix that the solution found in (a) is indeed a (constrained) maximum.

   (c) Assume, a second constraint, $4x_1-2x_2+7x_3=8$, is added to the above problem.
   - How does the bordered Hessian look like in this case?
   - Which principle minors of the Hessian would now have to be checked in (b) and what would be the corresponding criterion for a maximum? Only answer this question but don't carry through the corresponding computations!

3. (18 points: (a): 2, (b): 10, (c): 3, (d): 3) Consider the following second-order differential equation:

$$\ddot{x}(t) = -6\dot{x}(t) - 5x(t) + 15$$

(a) Transform the above equation into a system of two first-order equations.

(b) Solve the system obtained in (a) in the usual way.

(c) Check, that your solution determined in (b) does indeed fulfill the above second-order differential equation.

(d) Determine the special solution for the following initial values: $x(t=0) = -2$ and $x(t=1) = 3.68859$. (Unfortunately, the use of an electronic calculator is unavoidable here.)

4. (15 points: (a): 3, (b): 3, (c): 6, (d): 3) Consider the following system of two first-order differential equations:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -6 \\ 5 \end{pmatrix}.$$

(a) Compute the steady state of the system.

(b) Determine the type of dynamics (node/spiral/saddle point, stable/unstable etc.).

(c) Draw the zero-isoclines into the phase plane and determine the directions of motion above and below them.

(d) Draw (in a qualitative way) one or more (depending on the type of dynamics) typical trajectories into this picture.

5. (25 points: (a): 5, (b): 6, (c): 6, (d): 8) Assume, Robinson Crusoe wants to optimize his consumption path $c(t)$ over time according to the production technology $y(t) = [k(t)]^{1/2}$ (with y denoting output and k the capital stock). Concretely, he is interested in maximizing his lifetime utility

$$\begin{aligned} \mathcal{U} &= \int_0^\infty \ln(c(t))e^{-0.05t}dt \\ \text{subject to} \quad \dot{k}(t) &= [k(t)]^{1/2} - c(t) \quad \text{(thus, there is no depreciation of the capital stock)} \\ k(0) &= 98 \end{aligned}$$

(a) Set up the (present-value-) Hamiltonian function for this problem and derive the necessary conditions for an optimal path.

(b) Derive from the results obtained in (a) a dynamical system in the variables c and k and determine its steady state.

(c) Linearize the dynamical system determined in (b) at the steady state and verify with the aid of the Jacobian matrix, that the resulting dynamics is of the saddle point type.

(d) Compute the stable branch of the linearized system and assume, that this approximation is of sufficient quality also for the initial value $k(0) = 98$ mentioned above. Which consumption level $c(0)$ has to be chosen at $t = 0$ in order to reach the saddle path? (Again, the use of an electronic calculator is unavoidable here; sorry!)