Examination: Mathematical Economics **Lecture Number:** 1350
Examiner: Dr. G. Groh **Wintersemester 2002/2003**
Hint: 75 of the 100 points attainable are regarded as the maximum number one can reach in the time available.
The following aids can be used: Electronic calculator and dictionary
Examination questions:

1. (24 points: (a): 8, (b): 7, (c): 3, (d): 6) Consider the following nonlinear programming problem:

$$\begin{aligned} \max_{x_1,x_2} \quad & 6x_1 + 8x_2 \\ \text{subject to} \quad x_1^2 + x_2^2 & \leq 25 \\ 2x_1 + 5x_2 & \geq 10 \\ x_1, x_2 & \geq 0 \end{aligned}$$

   (a) Give a graphical representation of the two constraints, the resulting set of feasible solutions and the contour curves of the maximand function and try to identify the solution.

   (b) Set up the Kuhn-Tucker-conditions and show, that a solution containing $x_1 = 0$ or $x_2 = 0$ or $\lambda_1 = 0$ can directly be ruled out.

   (c) Now assume additionally $\lambda_2^* = 0$ and compute the resulting values for x_1^*, x_2^* and λ_1^*. Check, that they are compatible with <u>all</u> Kuhn-Tucker-conditions.

   (d) Show that the solution obtained in (c) fulfills the properties of a saddlepoint of the Lagrange-function. Which conclusions can be drawn from this fact?

2. (20 points: (a): 5, (b): 7, (c): 3, (d): 5) Consider the following linear programming problem:

$$\begin{aligned} \max_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad x_1 + 2x_2 & \leq 14 \\ 3x_1 + x_2 & \leq 12 \\ x_1, x_2 & \geq 0 \end{aligned}$$

   (a) Draw a graph containing the constraint set and the contour lines of the objective function and try to obtain the optimal solution.

   (b) Now solve the problem by means of the Simplex-algorithm (you can use the method of the lecture or a Simplex-tableau, as you like).

   (c) Now set up the corresponding dual problem.

   (d) Compute the solution of the dual problem, making use <u>only</u> of the duality theorem(s) and the primal's solution obtained in (a) or (b), respectively.

3. (16 points: (a): 8, (b): 3, (c): 5)

   (a) Determine the general solution of the following system of two linear differential equations:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ -18 & -5 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

See overleaf

(b) Compute the special solution for the following initial condition: $x_1(0) = 3; \quad x_2(0) = 2.$

(c) Derive from the system in (a) a single second-order differential equation in $x_1(t)$, that does not contain $x_2(t)$ or its derivative (i.e. determine $\ddot{x}_1(t)$ in dependence on $\dot{x}_1(t)$ and $x_1(t)$ only).

4. (20 points: (a): 4, (b): 4, (c): 3, (d): 6, (e): 3) Consider the following nonlinear system of differential equations:

$$\begin{aligned} \dot{x} &= \frac{1}{5}x^2 - \frac{2}{5}xy - \frac{3}{5}x \\ \dot{y} &= 3xy + y^2 - 16y \end{aligned}$$

(a) Determine that steady state of the above system, which is characterized by $x \neq 0$ and $y \neq 0$.

(b) Show, that the linearization of the system at the steady state determined in (a) yields:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \approx \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} -3 \\ -16 \end{pmatrix}.$$

(c) Determine the type of dynamics of the system in (b) <u>without</u> computing the explicit solution.

(d) Draw the zero-isoclines of the system in (b) into the phase plane and determine the directions of motion above and below them.

(e) Draw (in a qualitative way) one or more (depending on the type of dynamics) typical trajectories into this picture.

5. (20 points: (a): 5, (b): 6, (c): 5, (d): 4) Consider a firm, that wants to maximize the present value of its current and future cash flows by an appropriate choice of the time paths for labor input $N(t)$ and gross investment $I(t)$ ($K(t)$ thereby denoting the capital stock):

$$\max_{N(t),I(t)} \int_0^{\infty} \left[6[N(t)]^{\frac{1}{2}}[K(t)]^{\frac{1}{2}} - 1.5 \cdot N(t) - 5 \cdot [I(t)]^2\right] e^{-0.06t} dt$$

$$\begin{aligned} \text{subject to} \quad \dot{K}(t) &= I(t) - 0.04 \cdot K(t) \\ K(0) &= 130 \end{aligned}$$

(a) Set up the present-value-Hamiltonian function for this problem and derive the necessary conditions for an optimal path.

(b) Derive from the results obtained in (a) a two-dimensional system of differential equations in the variables K and I.

(c) Compute the explicit general solution of the system derived in (b). What type of dynamics does emerge?

(d) Now compute the concrete solution of the optimization problem for the given initial value $K(0) = 130$. In addition to this, determine also the values for $K(t = 10), I(t = 10)$ and $N(t = 10)$ (unfortunately, the use of the electronic calculator is now unavoidable).