Original

Examination:	Mathematical Economics	**Lecture Number:** 7350
Examiner:	Dr. G. Groh	**Summersemester 2004**
Hint:	75 of the 100 points attainable are regarded as the maximum number one can reach in the time available.	

The following aids can be used: Electronic calculator and dictionary

Examination questions:

1. (25 points: (a): 6, (b): 2, (c): 6, (d): 7, (e): 4) Consider the following nonlinear programming problem:

$$\begin{aligned} \max_{x_1, x_2} \quad & 25 - 7x_2 + 3x_1^3 - 9x_1^2 + 9x_1 \\ \text{subject to} \quad 6x_1 + x_2 - 18 &\leq 0 \\ x_1 - 8x_2^2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (a) Set up the Lagrange-function and the Kuhn-Tucker-conditions. Show, that solutions containing $x_2^* > 0$ can directly be ruled out (Hint: A sharp look at one of the Kuhn-Tucker-conditions already reveals this fact).

   (b) Now show, that additionally $x_1^* > 0$ and $\lambda_2^* = 0$ must hold true.

   (c) Compute the solutions of the Kuhn-Tucker-conditions (Hint: Consider the two cases $\lambda_1^* = 0$ and $\lambda_1^* > 0$ separately).

   (d) Check, whether the Kuhn-Tucker-conditions are necessary and/or sufficient for a global constrained maximum here.

   (e) Which additional information do you get from Weierstrass' theorem here? What does it imply together with your results from (d) for your solutions obtained in (c)?

2. (20 points: (a): 1, (b): 6, (c): 1, (d): 4, (e): 8) Consider the following price-adjustment equations for two markets with $\dot{p}_1(t)$ and $\dot{p}_2(t)$ depending on the excess demand for good 1 and good 2, respectively:

$$\dot{p}_1(t) = \alpha \left[\frac{1}{2} \cdot \frac{p_2(t)}{p_1(t)} + \frac{1}{2} \cdot \frac{1}{p_1(t)} - 1 \right], \quad \alpha > 0 \tag{1}$$

$$\dot{p}_2(t) = \beta \left[\frac{1}{2} \cdot \frac{p_1(t)}{p_2(t)} + \frac{1}{4} \cdot \frac{1}{p_2(t)} - \frac{3}{4} \right], \quad \beta > 0 \tag{2}$$

   (a) Compute the steady state $(\bar{p}_1, \bar{p}_2)$ for the above system.

   (b) Note, that the system (1), (2) can be rewritten as well in terms of the differences between the current price level and the corresponding steady state value: $(p_i - \bar{p}_i)$, $i = 1, 2$. Since $\frac{d}{dt}[p_i - \bar{p}_i] \equiv \dot{p}_i$ $(i = 1, 2)$, the right hand sides of (1) and (2) remain unchanged. Now show, that

$$V\left((p_1 - \bar{p}_1), (p_2 - \bar{p}_2)\right) := \tfrac{1}{2\alpha}(p_1 - \bar{p}_1)^2 + \tfrac{1}{2\beta}(p_2 - \bar{p}_2)^2$$

See overleaf

fulfills all requirements of a Liapunov-function in the two variables $(p_1 - \bar{p}_1)$ and $(p_2 - \bar{p}_2)$.
Hint: You can use

$$p_i + \frac{1}{p_i}\left\{ \begin{array}{ll} = 2 & \text{for} \quad p_i = 1 \\ > 2 & \text{for} \quad 0 < p_i \neq 1 \end{array} \right\}, \quad i = 1,2 \quad \text{and} \quad \frac{p_2}{p_1} + \frac{p_1}{p_2} \geq 2 \quad \text{for} \quad p_1, p_2 > 0. \tag{3}$$

(c) Which conclusion can be drawn from the result in (b)?

(d) Derive from the original system (1), (2) the zero-isoclines for p_1 and p_2 and draw them into the phase plane. Determine the directions of motion above and below them.

(e) Now linearize the system (1), (2) at the steady state and determine the type of the dynamics in a neighborhood around it without computing the explicit solution.

3. (15 points: (a): 5, (b): 5, (c): 5)

(a) Compute the general solution of the following linear differential equation:

$$\dot{x}(t) = -2.5x(t) + 5t + 4.5$$

(b) Show that the following differential equation is an exact one and solve it in the corresponding way:

$$\dot{x}(t) \cdot (t^2 + 3) + 2x(t) \cdot t + 5 = 0$$

(c) Check the Routh-Hurwitz-conditions for the following linear system of differential equations:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 2 & -4 \\ 5 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

Which conclusion can be drawn from the results?

4. (17 points: (a): 5, (b): 3, (c): 1, (d): 3, (e): 5) Consider the following system of two linear difference equations:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} 1 - 2\sqrt{3} \\ 3 - \sqrt{3} \end{pmatrix}.$$

(a) Verify, that $(\bar{x}, \bar{y}) = (2, 1)$ is the steady state of the system. Then determine the eigenvalues and the eigenvectors of the system and set up the solution in a straightforward way.

(b) Now express the two conjugate complex eigenvalues in terms of their polar coordinates: $a \pm ib = r(\cos\theta \pm i \sin\theta)$, with $r := \sqrt{a^2 + b^2}$ and θ being the angle enclosed ($\sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$). For the determination of θ you can use the following tables:

See next page

$\sin\theta$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
θ	$\frac{1}{6}\pi(30°)$ or $\frac{5}{6}\pi(150°)$	$\frac{1}{4}\pi(45°)$ or $\frac{3}{4}\pi(135°)$	$\frac{1}{3}\pi(60°)$ or $\frac{2}{3}\pi(120°)$

$\cos\theta$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
θ	$\frac{1}{3}\pi(60°)$ or $\frac{5}{3}\pi(300°)$	$\frac{1}{4}\pi(45°)$ or $\frac{7}{4}\pi(315°)$	$\frac{1}{6}\pi(30°)$ or $\frac{11}{6}\pi(330°)$

(c) Now recall deMoivre's theorem, according to which $r^t(\cos\theta \pm i\sin\theta)^t = r^t(\cos(\theta t) \pm i\sin(\theta t))$ and carry through the corresponding changes in your solution. Rearrange the terms in a suitable way.

(d) Finally, choose for the two parameters of the solution two conjugate complex numbers and show, that now an entirely real-valued solution results.

(e) Let the following initial values be given: $x_0 = 8; y_0 = 5$. Compute the resulting values for x_6 and y_6.

5. (23 points: (a): 6, (b): 3, (c): 10, (d): 4)

Assume, a social planner tries to choose per-capita-consumption $c(t)$ and the provision of a public good $G(t)$ such as to maximize the welfare of a representative consumer. Output is produced by an aggregate production function $Y(t) = \gamma[K(t)]^{\frac{1}{3}}N^{\frac{2}{3}}$ with $K(t)$ denoting the total capital stock and N the population size. Capital accumulation is then determined by the difference between $Y(t)$ and the resources devoted to private ($Nc(t)$) and public consumption ($G(t)$) (the depreciation rate is assumed to be equal to zero):

$$\max_{c(t),G(t)} \int_0^\infty [40000\ln(c(t)) + 8000\ln(G(t))]\, e^{-0.1t}dt$$

$$\begin{aligned} \text{subject to} \quad \dot{K}(t) &= 1.2[K(t)]^{\frac{1}{3}}N^{\frac{2}{3}} - N\cdot c(t) - G(t), \quad N = 1000 \\ K(0) &= 8300 \end{aligned}$$

with the state variable K and the two control variables c and G.

(a) Set up the current-value-Hamiltonian function and derive the necessary conditions for an optimal path.

(b) Derive from the results obtained in (a) a two-dimensional system of differential equations in the state-variable K and the costate variable $\tilde{\lambda}$ and determine its steady state.

(c) Linearize the dynamical system determined in (b) at the steady state and compute the explicit general solution.

(d) Compute the values for $c(0)$ and $G(0)$ on the basis of the linearized system obtained in (c). Don't worry, if the resulting numbers are no integers.