HA | HS

Examination:	Mathematical Economics	**Lecture Number:** 1798
Exam:	Dr. G. Groh	**Wintersemester 2004/2005**
Hint:	75 of the 100 points attainable are regarded as the maximum number one can reach in the time available.	

The following aids can be used: Electronic calculator and dictionary

Examination questions:

1. (18 points: (a): 7, (b): 6, (c): 5)
   Assume, a firm has 14 units of a good y at its disposal, that can be sold on two separated markets. The demand on the first market is given by $y_1(p_1) = -p_1 + 18$ and that on the second market by $y_2(p_2) = -2p_2 + 16$. The firm's problem is now to choose the two prices p_1 and p_2 in such a way that total revenues $R := p_1 y_1 + p_2 y_2$ are maximized:

   $$\max_{p_1,p_2} R(p_1,p_2) = p_1 \cdot y_1(p_1) + p_2 \cdot y_2(p_2) \quad \text{s.t.:} \quad y_1(p_1) + y_2(p_2) \;=\; 14$$

   (Assume, that y is a perishable good and that due to extremely high storage costs the whole quantity of 14 units has to be sold. Thus, the equality sign in the constraint can be justified. Furthermore, nonnegativity constraints for the prices are not taken into account here.)

   (a) Plug in the demand functions, set up the Lagrange function and compute the solution via the corresponding first-order-conditions of classical programming.

   (b) Check with the aid of the bordered Hessian matrix that the solution found in (a) is indeed a (local) constrained maximum.

   (c) Now determine only with the aid of the Lagrange-function and the envelope theorem the changes of the maximum revenue that emerge, if there is a shift in one of the demand curves or a change in the quantity of the good at the firm's disposal. Thus, reformulate the above maximization problem with $y_1 = -p_1 + a$, $\quad y_2 = -2p_2 + b$ and $y_1 + y_2 = \bar{y}$ and determine $\frac{\partial R}{\partial a}(p_1^*(a), p_2^*(a), \lambda^*(a))$, $\quad \frac{\partial R}{\partial b}(p_1^*(b), p_2^*(b), \lambda^*(b))$ and $\frac{\partial R}{\partial \bar{y}}(p_1^*(\bar{y}), p_2^*(\bar{y}), \lambda^*(\bar{y}))$. After having done this, compute the numerical values of these three derivatives for the original problem, i.e. for $a = 18, b = 16$ and $\bar{y} = 14$.

2. (20 points: (a): 4, (b): 1, (c): 6, (d): 5, (e): 4)
   Consider the following homogeneous system of two linear differential equations:

   $$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} \;=\; \begin{pmatrix} 5 & -4 \\ 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

   (a) Compute the eigenvalues of the system's matrix (they should be complex) and the corresponding eigenvectors (which should contain complex components, too).

   (b) Set up the general solution in a straightforward way by directly using the results from (a).

   (c) Now recall the general relationship $e^{\pm i\beta t} = \cos(\beta t) \pm i \sin(\beta t)$ and make use of it in your solution from (b). Having done this, rearrange the terms in a meaningful way.

   (d) Show that all imaginary parts can be eliminated, if the two parameters in your solution are chosen in an appropriate way.

   (e) Finally, assume the following initial values: $x_1(0) = 28$ and $x_2(0) = 2$. Determine on this basis the values for the parameters of the solution.

– See overleaf –

3. (20 points: (a): 6, (b): 6, (c): 3, (d): 4, (e): 1) Consider the following nonlinear dynamical system:

$$\begin{aligned}\dot{x}(t) &= -10x(t) - 2[x(t)]^3 - 4x(t)[y(t)]^2 - 2y(t)\\ \dot{y}(t) &= -y(t) - 7[y(t)]^3 - 12[x(t)]^2 y(t) + 3x(t)\end{aligned}$$

(a) Linearize the system at the steady state $(\bar{x}, \bar{y}) = (0, 0)$ and determine the type of the dynamics without computing the explicit solution.

(b) Draw the zero-isoclines of the system in (a) into the phase plane and determine the directions of motion above and below them.

(c) Draw (in a qualitative way) one or more (depending on the type of dynamics) typical trajectories into this picture.

(d) Now consider the system again in its original nonlinear form. Show, that a function of the form $V(x, y) = ax^2 + by^2$ fulfills all requirements for a Liapunov-function, if the coefficients a and b are chosen in a suitable way.

(e) Which conclusion can be drawn from the existence of such a Liapunov-function?

4. (17 points: (a): 1, (b): 7, (c): 4, (d): 5) Consider the following second-order linear difference equation:

$$x_{t+2} = 2x_{t+1} + 8x_t - 27$$

(a) Transform the equation into a first-order system in two dynamic variables.

(b) Compute the explicit general solution for the system of (a).

(c) Show, that your solution from (b) indeed fulfills the above second-order equation.

(d) Now assume $x_{t=2} = 111$ and $x_{t=3} = 267$ and determine on this basis the value for $x_{t=5}$.

5. (25 points: (a): 5, (b): 3, (c): 6, (d): 4, (e): 7) A firm is going to be liquidated at T=30. Up to this time, the present value of the cash flows shall be maximized by an appropriate choice of the time paths for labor input $L(t)$ and gross investment $I(t)$ ($K(t)$ thereby denoting the capital stock). At the end of the time horizon $T = 30$ the remaining capital stock can be sold at a price of 8 per unit of capital. Given a production function $F(L(t), K(t)) = 12[L(t)]^{\frac{1}{2}}[K(t)]^{\frac{1}{2}}$, a real wage rate of 3, investment costs of $5[I(t)]^2$, a real rate of interest of 6 % , a depreciation rate of 4 % and an initial capital stock of 320, the problem can be formulated as follows:

$$\max_{L(t),I(t)} \int_0^T \left[12[L(t)]^{\frac{1}{2}}[K(t)]^{\frac{1}{2}} - 3 \cdot L(t) - 5 \cdot [I(t)]^2\right] e^{-0.06t} dt + 8K(T)e^{-0.06T} \quad \text{with} \quad T = 30$$

$$\begin{aligned}\text{subject to} \quad \dot{K}(t) &= I(t) - 0.04 \cdot K(t)\\ K(0) &= 320\end{aligned}$$

(a) Set up the current-value-Hamiltonian function for this problem and derive the necessary conditions for an optimal path.

(b) Derive from the results obtained in (a) a two-dimensional system of differential equations in the state-variable K and the costate-variable λ.

(c) Compute the explicit general solution of the system derived in (b). What type of dynamics does emerge?

(d) Now compute the concrete solution of the optimization problem for the given initial value $K(0) = 320$. Having done so, determine the optimal time paths for the two control variables $L(t)$ and $I(t)$, too. What are the values for $L(t = 0), I(t = 0)$ and $K(T = 30)$?

(e) Draw a phase diagram for the system in (b) (with zero-isoclines, arrows and trajectories). In which region is the optimal trajectory located?