Examination: Mathematical Economics **Lecture Number:** 1798
Examiner: Dr. G. Groh **Summersemester 2005**
Hint: 75 of the 100 points attainable are regarded as the maximum number one can reach in the time available.
The following aids can be used: Electronic calculator and dictionary

Examination questions:

1. (29 points: (a): 11, (b): 2, (c): 4, (d): 6, (e): 6) Have a look at the following cost minimization problem with x_1, x_2, x_3 being the quantities of the input goods and $F(x_1, x_2, x_3) = 0.25 \cdot x_1 x_2 x_3$ denoting a production function (with increasing economies of scale):

$$\begin{array}{rrcll} \min\limits_{x_1,x_2,x_3} & 40x_1 + 90x_2 + 72x_3 & & & \\ \text{subject to} & F(x_1,x_2,x_3) = 0.25 \cdot x_1 x_2 x_3 & \geq & 360 & \text{(minimum production constraint)} \\ & 19x_1 & \leq & 152 & \text{(capacity constraint)} \\ & x_1, x_2, x_3 & \geq & 0 & \end{array}$$

   (a) Reformulate the problem in standard form, set up the Lagrange-function and derive from it the Kuhn-Tucker-conditions.

   (b) Show by a straightforward arguement, that $x_i \neq 0, \quad i = 1, 2, 3$. Having done this, show that $\lambda_1 = 0$ would immediately lead to a contradiction (with λ_1 as the Lagrange-multiplier for the first constraint).

   (c) Now assume $\lambda_2 = 0$ and derive a contradiction as well. (Here some computations have to be done.)

   (d) Use the results from (b) and (c) and compute the solution.

   (e) Check, whether the Kuhn-Tucker-conditions are necessary and/or sufficient for an optimal solution here. (<u>Hint</u>: In conjunction with the considerations to be made here you can directly use the fact, that the upper contour set of the first constraint function is convex.)

2. (13 points: (a): 5, (b): 6, (c): 2) Consider the following two-dimensional system of differential equations:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 4 & -8 \\ 12 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 8 \\ -60 \end{pmatrix}.$$

   (a) Compute the steady state and determine the type of the resulting dynamics <u>without computing the explicit solution.</u>

   (b) Draw the zero-isoclines into the phase plane and determine the directions of motion above and below them.

   (c) Now draw one or more (depending on the type of dynamics) typical trajectories into this picture.

See overleaf

3. (21 points: (a): 3, (b): 7, (c): 11) Assume an economy consisting of three industries. Each industry i needs c_{ii} units of its own production and c_{ji} units of industry j's output for the production of one unit of good i. With d_i as the final demand for this good and $x_i(t)$ as industry i's total supply, the excess demand for this product can be written as

$$d_i + \left[\sum_{j=1}^{3} c_{ij}x_j(t)\right] - x_i(t)$$

If $x_i(t)$ now adjusts to this excess demand with an adjustment speed of 4, one immediately gets:

$$\dot{x}_i(t) = 4\left[(c_{ii}-1)x_i(t) + \left(\sum_{j\neq i} c_{ij}x_j(t)\right) + d_i\right], \qquad i = 1,2,3$$

Let now the matrix C of input coefficients be

$$C = \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.25 & 0.5 & 0 \\ 0.5 & 0.25 & 0.25 \end{pmatrix} \quad \text{and the final demand vector} \quad d = \begin{pmatrix} 5 \\ 5 \\ 15 \end{pmatrix}.$$

(a) Set up the resulting three-dimensional system of differential equations.

(b) Check by means of the Routh-Hurwitz conditions, whether the steady state of this system is stable.

(c) Now compute the explicit general solution. (Hint: Recall that the general solution for a system $\dot{x}(t) = Ax(t) + b, \quad x(t), b \in I\!R^3$ can be written as $\underbrace{x(t)}_{(3\times1)} = \underbrace{\bar{x}}_{(3\times1)} + \underbrace{\tilde{P}}_{(3\times3)} \cdot \underbrace{e^{\tilde{\Lambda}t}}_{(3\times3)} \cdot \underbrace{\gamma}_{(3\times1)}$ with $A \cdot \tilde{P} = \tilde{P} \cdot \tilde{\Lambda}$.)

4. (17 points: (a): 1, (b): 1, (c): 4, (d): 4, (e): 4, (f): 3) Consider the following nonlinear system of difference equations:

$$\begin{aligned} x_{t+1} &= \sin^2(x_t) + \cos^2(y_t) - 1.5x_ty_t - 1.5x_t + 4y_t - 1 \\ y_{t+1} &= y_te^{x_t} - x_te^{y_t} + 0.5x_t + 0.5y_t \end{aligned}$$

(a) Verify, that $(\bar{x}, \bar{y}) = (0, 0)$ is a steady state of this system.

(b) Try to find another steady state. (Hint: Recall $\sin^2(x) + \cos^2(x) = 1 \quad \forall x \in I\!R$.)

(c) Set up the Jacobian matrix and evaluate it then at the steady state $(0, 0)$.

(d) Compute the explicit general solution for the linearized system.

(e) Determine on this basis the values for $x_{t=4}$ and $y_{t=4}$, if $x_{t=3} = 6$ and $y_{t=1} = 16$.

(f) Is it possible to find a Liapunov-function $V(x_t, y_t)$ for the original nonlinear system which satisfies

   i. $V(x_t, y_t) > 0 \quad \forall (x_t, y_t) \neq (0,0)$ and $V(0,0) = 0$

   ii. $V(x_t, y_t) \to +\infty$ as $\|(x_t, y_t)\| \to +\infty$

   iii. $V(x_{t+1}, y_{t+1}) - V(x_t, y_t) < 0 \quad \forall (x_t, y_t) \in I\!R^2 \backslash \{(0,0)\}$ and $V(x_{t+1}, y_{t+1}) - V(x_t, y_t) = 0$ for $(0,0)$?

   Try to answer the question without any computations!

See next page

5. (20 points: (a): 5, (b): 3, (c): 4, (d): 5, (e): 3)

A firm wants to maximize the present value of future profits by an appropriate choice of the time paths of output $x(t)$ and marketing efforts $a(t)$. Its inverse demand function depends on output x and the goodwillstock A in the following way:

$$p(x, A) = 2A^{0.2}x^{-0.5} \quad \text{(with } p \text{ denoting the output price)} \quad \Longrightarrow \quad p \cdot x = 2A^{0.2}x^{0.5}$$

With $\frac{1}{6}$ being the production costs per unit of output, $\frac{1}{8}a^2$ the costs for marketing activities of size a and $\dot{A} = a - \delta A$ describing the relation between the latter and the goodwillstock A, the problem can be summarized as follows (given a rate of interest of 0.1, an initial value $A(0) = 30$ and $\delta = 0.15$):

$$\max_{x(t),a(t)} \int_0^{\infty} \left[2[A(t)]^{0.2}[x(t)]^{0.5} - \frac{1}{6}x(t) - \frac{1}{8}[a(t)]^2\right] e^{-0.1t}dt$$

$$\text{subject to} \quad \begin{aligned} \dot{A}(t) &= a(t) - 0.15A(t) \\ A(0) &= 30 \end{aligned}$$

(a) Set up the current-value-Hamiltonian function and derive the necessary conditions for an optimal path.

(b) Derive from the results obtained in (a) a two-dimensional system of differential equations in the two variables $A(t)$ and $a(t)$.

(c) Linearize the dynamical system determined in (b) at the steady state.

(d) Compute the explicit general solution of the system derived in (c).

(e) Determine the concrete solution for the given initial value $A(0) = 30$ and compute the values for $x(0)$, $a(0)$ and $A(10)$. (Assume that the linearization is still valid in this region.)