Final Exam: 1798 Mathematical Economics, Winter 2005/06

No aids permitted except for (1) language dictionaries without any marking, (2) non-programmable pocket calculators without communicating and/or data processing functions, and (3) the hand out of the collection of formulas.

There are five problems on this exam. Solve all of them.

1. (30 points) Consider the following nonlinear programming problem where $x_1, x_2 \in [0, \infty)$:

$$\max_{x_1,x_2} \quad 2\sqrt{x_1 x_2}$$

$$\text{subject to}$$

$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ x_1 + 2x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (a) (8) Provide a graphical representation of (i) the constraint set that contains all feasible solutions and (ii) the contour curves of the objective function. Solve the maximization problem geometrically.

   (b) (2) Does the Weierstrass theorem ensure that a solution to this problem exist?

   (c) (2) Provide a direct argument why $x_1 = 0$ or $x_2 = 0$ cannot be part of the solution.

   (d) (6) Are the Kuhn-Tucker-conditions necessary and/or sufficient for a local/global optimum. Would this optimum be unique?

   (e) (8) Set up the Lagrangian and the Kuhn-Tucker-conditions. Find all candidates that satisfy the Kuhn-Tucker-conditions.

   (f) (4) What is the optimum? Verify that the the proposed optimum candidate satisfies all Kuhn-Tucker conditions.

2. (30 points) Consider a government that chooses tax revenue T and public expenditure G to maximize social welfare. The government's social welfare criterion is represented by the twice-continuously differentiable function $W(T, G)$. The government has to satisfy two constraints: Firstly, it is not allowed to run a budget deficit larger than Δ which is an exogenous parameter. Secondly, it has to guarantee a minimum level of public service that requires public expenditure of $G_o > 0$ implying that feasible choices (T, G) satisfy $G \geq G_o$. Assume that $0 \leq \Delta < G_o$. The social welfare function $W : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfies the following properties for all $T, G \in [0, \infty)$:

$$\frac{\partial W(T,G)}{\partial T} < 0, \; \frac{\partial^2 W(T,G)}{\partial T^2} < 0, \; \frac{\partial W(T,G)}{\partial G} > 0, \; \frac{\partial^2 W(T,G)}{\partial G^2} < 0, \; \frac{\partial^2 W(T,G)}{\partial G \partial T} = 0$$

and the maximzation problem of the government is given by:

$$\max_{T,G} \qquad W(T,G)$$

$$\text{subject to}$$

$$\begin{aligned} \Delta + T &\geq G \\ G &\geq G_o \end{aligned}$$

   (a) (2) Briefly indicate why it is trivial that feasibility requires that government expenditures and tax revenue have to be strictly positive, i.e. $G > 0$ and $T > 0$.

(b) (10) Obtain a geometric solution of the problem in T-G-space (T on horizontal axis, G on vertical axis) using the implicit function theorem to characterize the shape (slope and convexity/concavity) of the contour curves of the objective function. Distinguish between two qualitatively different solutions, one where both constraints are active and another one where the number of active constraints is smaller than 2. Provide a geometric sketch for each type of solution.

(c) (3) Set up the Lagrangian and give the Kuhn-Tucker conditions assuming that $G, T > 0$.

(d) (15) Solve the problem analytically:

   (i) Show that the government budget constraint is always active.

   (ii) Characterize both qualitatively different solutions by relating marginal welfare effects of government expenditure and tax revenue in the optimum to one another, i.e. find a condition that relates $\partial W(T^*, G^*)/\partial G$ to $\partial W(T^*, G^*)/\partial T$ in each case; call it optimality condition. [Hint: In the case of two active constraints, eliminate one Lagrangian multiplier and use that fact that Lagrangian multipliers are required to be nonnegative.]

   (iii) Briefly interpret the identifed optimality condition for each case.

3. (15 points) Suppose that two interdependent markets jointly evolve over time, however, the steady state of the underlying dynamic system is characterized by the following system of equations:

$$\begin{aligned} p_1 - p_2 - 100 - 2\delta &= 0. \\ 2\delta - 50 + 2p_2 &= 0. \end{aligned}$$

(a) (2) Write the system in matrix notation $Ap = b$ where A is the matrix of coefficients, p is the vector of prices and b is 2×1.

(b) (8) Does there exist a solution to the linear system? If yes, solve it using matrix algebra, i.e. find $p = A^{-1}b$.

(c) (5) Use the implicit function theorem to determine the effects of infinitesimally small variations of δ on steady state prices.

4. (15 points) Consider the following nonlinear difference equation of first-order:

$$x_t - 0.5x_{t-1} - \sqrt{x_{t-1}} = 0.$$

(a) (3) Find all steady states of the given difference equation.

(b) (4) Geometrically determine the stability property of each steady state.

(c) (3) Verify that linearizing the difference equation around the nontrivial steady state leads to

$$x_t - 0.75x_{t-1} - 1 = 0.$$

(d) (5) Find the definite solution for the linearized difference equation given the initial condition $x(0) = 1$. Briefly describe the behavior of x_t over the long-run, i.e. as time tends to infinity.

5. (10 points) Consider the following nonlinear differential equation:

$$\dot{y} = -a \cdot y + b(t)$$

(a) (4) Assume for this part of the problem that the forcing function is a constant, specifically $b(t) \equiv 1$. Use a phase diagram (y-$\dot{y}$-space) to determine the stability properties of the steady state if (i) $a = -2$ and (ii) $a = 2$.

(b) (6) Find the general solution of the differential equation where $a \in \mathbb{R}$ and $b = 5t + 2$ and verify it.