Exam: Methods for Economists (20312)
Examiner: Prof. Dr. Marco Runkel **Winter 2010/11**

Remarks

i. The following aids can be used: bilingual dictionary, calculator according to the examination office's list, one sheet of paper with formulas written on both sides (either handwritten or typed).

ii. The exam consists of **four problems**, which **all** have to be answered.

iii. Total time available is **120 minutes**.

iv. Please write readable and leave a margin at the right for corrections.

v. The questions can be answered either in English or in German.

Problem 1 (25 ponits)

Consider the macroeconomic IS-model with the following equilibrium for the goods-market:

$$
\begin{aligned}
Y &= C + \bar{I} + \bar{G} \\
C &= C_0 + c(Y - T), \qquad 0 < c < 1 \\
T &= T_0 + tY, \qquad 0 < t < 1
\end{aligned}
$$

where Y is income, C consumption, $\bar{I}$ investment, $\bar{G}$ government expenditures and T taxes. Autonomous consumption is denoted by C_0 and the marginal pronpensity to consume by c. Taxation T consists of lump sum tax T_0 and an income tax t.

a) Apply the implicit function theorem to this set of equations (**without** reducing it to one single equation before) and determine the expressions for $\frac{dY}{d\bar{G}}$, $\frac{dC}{d\bar{G}}$ and $\frac{dT}{d\bar{G}}$. Give a comprehensive intuition for your results. (15 points)

b) Assume that the economy under consideration opens to trade with the rest of the world and its new goods market equilibrium is described by the following equations:

$$
\begin{aligned}
Y &= C + \bar{I} + \bar{G} + \bar{EX} - mY, \qquad 0 < m < 1 \\
C &= C_0 + c(Y - T), \qquad 0 < c < 1 \\
T &= T_0 + tY + \tau mY, \qquad 0 < t < 1, 0 < \tau < 1
\end{aligned}
$$

where $\bar{EX}$ is exogenous export, mY import of foreign goods, m the marginal propensity to import and τ a tariff on imported goods. Determine $\frac{dY}{d\bar{G}}$ in the open economy and compare to your previous result. Explain the differences which arise. (10 points)

Please turn over!

Problem 2 (25 points)
Consider a two-good world, where both goods x and y are rationed. The individual's utility function is given by $U(x,y)$. Her total budget is denoted by B and the prices of the two goods are p_x and p_y, respectivelly. Furthermore, because of the rationing of the goods, the individual has coupons in the amount of C, which can be used to purchase x and y at prices c_x and c_y. Thus, the individual faces the following problem:

$$\max_{x,y} U(x,y) \quad \text{subject to} \quad \begin{aligned} B &\geq p_x x + p_y y \\ C &\geq c_x x + c_y y \end{aligned}$$

a) Set up the Lagrangian and write down the Kuhn-Tucker conditions. (5 points)

b) Now assume that $U = xy^2$, $B = 100$, $p_x = p_y = 1$, $C = 120$, $c_x = 2$ and $c_y = 1$. Show that $(x,y) = (0,0)$ cannot be an optimum even though it satisfies the Kuhn-Tucker conditions. (5 points)

c) Compute the optimal values of x, y, λ_1 and λ_2. (10 points)

d) Are the Kuhn-Tucker conditions necessary and/or sufficient for a maximum? (5 points)

Problem 3 (25 points)
Consider the following system of nonlinear differential equations:

$$\begin{aligned} \dot{x}(t) &= -6x + 5xy^3 - 6x^2y + y \\ \dot{y}(t) &= -3x - x^4 - 2y \end{aligned}$$

a) Show that the linearization of this system around the equilibrium point $(\bar{x}, \bar{y}) = (0,0)$ is given by:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -6 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$ (5 points)

b) Compute the roots of the characteristic equation and write down the general solution of the linearized system. (10 points)

c) Draw a phase diagram, including the demarcation lines and possible trajectories. Of what type is the equilibrium? (5 points)

d) Evaluate the solution for the initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$. (5 points)

Please turn over!

Problem 4 (25 points)

Assume that a benevolent government chooses the share of output to be devoted to provision of public goods $g, 0 < g < 1$ in order to maximize the lifetime utility of its resident. Income in this economy is a linear function of capital $Y = AK$, where Y is income, K capital and A a positive constant. Therefore, $G = gAK$ is the amount of government expenditures. Additionally, private consumption C is assumed to be constant. As a result, the problem of the government is given by:

$$\max_g U = \int_0^T (\ln C + \beta \ln G) e^{-\rho t} dt, \quad \beta > 0, \rho > 0$$

subject to the following constraints:

$$\begin{aligned} \dot{K} &= AK - C - G \\ G &= gAK \\ K(0) &= K_0, \quad K(T) \geq 0 \end{aligned}$$

a) Write down the current value Hamiltonian and the conditions for maximum. (5 points)

b) Compute the solution of the system in $a)$ and give the time paths of $K(t), G(t)$ and of the Lagrange multiplier. (10 points)

c) Which parameters determine whether the marginal utility of public good consumption is increasing over time? Explain your result. (5 points)

d) Calculate the indirect lifetime utility function $V(\beta, \rho, A, K_0, C)$. (5 points)