Remarks

i. The following aids can be used: bilingual dictionary, calculator according to the examination office's list, one sheet of paper with formulas written on both sides (either handwritten or typed).

ii. The exam consists of **four problems**, which **all** have to be answered.

iii. Total time available is **120 minutes**.

iv. Please write readable and leave a margin at the right for corrections.

v. The questions can be answered either in English or in German.

Problem 1 (25 points)

Consider a firm, which uses the production function $f(k) = ak - (b/2)k^2, a > 0, b > 0, k \in (0, b/a)$, where capital, denoted by k, is the sole factor of production. The firm can employ capital at the rental rate r. Its objective is to maximize its profit:

$$\Pi = pf(k) - rk,$$

where p denotes the market price of the product of the firm.

a) Find the first-order conditions for profit maximization. Apply the implicit function theorem to your result in order to find how the optimal quantity k^* reacts to changes in the price of capital r and the price of the final good p. Give a comprehensive intuition for your results. (15 points)

b) Use the envelope theorem to show how the optimal profit changes following an increase in the rental price of capital r. Explain your result. (10 points)

Problem 2 (25 points)
Consider the following utility maximization problem:

$$\max_{x,y} U(x,y) = x^2 + x + 4y^2 \quad \text{subject to} \quad 1 \geq 2x + 2y, \quad x \geq 0, \quad y \geq 0$$

where x and y denote the quantities of each good consumed.

a) Set up the Lagrangian and write down the Kuhn-Tucker conditions. (5 points)

b) Find all points, which satisfy the Kuhn-Tucker conditions. Find the optimal values of x, y and λ. (10 points)

c) Are the Kuhn-Tucker conditions necessary and/or sufficient for a maximum? (5 points)

d) Interpret the meaning of the lagrange multiplier. (5 points)

Please turn over!

Problem 3 (25 points)
Consider the following system of differential equations:

$$\begin{aligned} \dot{x}(t) &= -2x - 4y + 1 \\ \dot{y}(t) &= -x + y \end{aligned}$$

a) Find the equilibrium point $(\bar{x}, \bar{y})$ of the system. (5 points)

b) Compute the general solution of the system. (10 points)

c) Draw a phase diagram, including the demarcation lines and possible trajectories. Of what type is the equilibrium? (5 points)

d) Evaluate the solution for the initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (5 points)

Problem 4 (25 points)

Assume that a benevolent government wants to optimize the path of per-capita consumption c of the population over time in order to maximize the present value of welfare:

$$U = \int_0^T (15 \ln c) e^{-0.06t} dt,$$

subject to the constraints:

$$\begin{aligned} \dot{k} &= 3.2k^{0.25} - \delta k - c, \qquad \delta = 0.04 \\ k(0) &= k_0, \quad k(T) \geq 0 \end{aligned}$$

a) Set up the current value Hamiltonian for this problem and derive the necessary conditions for an optimum. (5 points)

b) Derive from the results obtained in *a*) a dynamical system in the variables k and c and determine their equilibrium values. (10 points)

c) Draw a phase diagram, including the demarcation lines and possible trajectories. Of what type is the equilibrium? (10 points)