Remarks:

1. The exam comprises 6 problems on 2 pages. You have to solve all of them.
2. Allowed aids: bilingual dictionary and pocket calculator according to the rules of the examination office.
3. Your time budget is 120 minutes.
4. When answering the questions below it is necessary that explanations and intermediate steps are provided!
5. Good luck!

Problem 1: (20 min)
Consider the following system of equations:

$$x^2 + y^2 = u^2 + v^2, \quad (1)$$

$$x + y = u - v. \quad (2)$$

Note that $x = -1, y = 1, u = 1, v = 1$ solves that system. All parts of this question refer to this point.

a) Calculate the derivatives:

$$\frac{dx}{du} = ..., \frac{dx}{dv} = ..., \frac{dy}{du} = ..., \frac{dy}{dv} = ... \quad (3)$$

b) Suppose that u and v simultaneously increase by 0.01. Provide an approximation of the change of x and y.

Problem 2 (15 min):
Consider the simple macroeconomic IS-MP which is described by the following set of equations:

$$Y = C(Y) + I(r) + G, \text{ with } G = \overline{G}, \qquad \text{(IS curve (IS))}$$

$$r = 2 + \alpha[Y - \overline{Y}], \text{ with } \alpha \geq 0, \qquad \text{(Taylor rule (TR))},$$

where Y is national income, C is consumption and $\overline{G}$ is exogenous government spending. The interest rate is denoted by r. $\overline{Y}$ is the potential output. Marginal consumption and investment have the properties: $0 < C'(Y) < 1,\ I'(r) < 0$.

a) Calculate the slope $\frac{dr}{dY}$ of the IS-curve (IS).

b) **Use the Implicit Function Theorem** to calculate the derivative $\frac{dY}{dG}$, i.e. do not simply substitute the second equation into the first.
Hint: You might need *Cramer's Rule*: Given a system of n linear equations and n unknowns, represented in matrix multiplication form as follows $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, where the n by n matrix $\mathbf{A}$ has a nonzero determinant, and the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is the column vector of the variables, there is a unique solution that can be computed by $x_i = \frac{|\mathbf{A_i}|}{|\mathbf{A}|}$, where $\mathbf{A_i}$ is formed by replacing the i^{th} row of $\mathbf{A}$ by $\mathbf{b}$.

Problem 3: (15 min)
Consider the following differential equation:

$$\dot{U} = s(L - U) - fU, \quad (4)$$

that describes the change in unemployment, where L (labour force), s (termination rate), f (hiring rate) > 0 are positive constants. Define the rate of unemployment as $u = U/L$.

a) Derive a differential equation for u.

b) Calculate the steady state u^*.

c) Draw a phase diagram and use it to discuss whether the steady state is stable or not.

d) Find the general solution of the differential equation.

Problem 4: (20 min)
Consider the following differential equation:

$$\dot{y}(t) = y(t)[1 - 2y(t)]. \tag{5}$$

a) Find the steady states of $y(t)$.

b) Draw a slope diagram.

c) Find the solution to the initial value problem:

c 1) with $y(0) = 1$,

c 2) with $y(0) = 1/4$.

Hint: For integration you will need the fact that there are constants A and B such that:

$$\frac{1}{y(1-2y)} = \frac{A}{y} + \frac{B}{1-2y}. \tag{6}$$

This is called *method of partial fractions*.

Problem 5: (25 min)

a) Find the solution of the following maximization problem:

$$\max_{x,y} \quad u_a = y + \frac{1}{3}x \tag{7}$$
$$\text{s.t.} \quad x \geq 0 \tag{8}$$
$$y \geq 0 \tag{9}$$
$$10 = x^2 + y^2 \tag{10}$$

b) Find the solution of the following maximization problem:

$$\max_{x,y} \quad u_b = y + \frac{1}{3}x \tag{11}$$
$$\text{s.t.} \quad x \geq 0 \tag{12}$$
$$y \geq 0 \tag{13}$$
$$y = \sqrt{1-x} \tag{14}$$

c) Draw a diagram visualizing the constraints and the level sets of the objective and the solutions. Compare the solutions.

Problem 6: (25 min)
Solve the following optimal control problem where $c(t)$ is the *control* and $s(t)$ is the *state*:

$$\max_{c(t)} \quad J = \int_0^1 \ln\left[4 \cdot c(t) \cdot s(t)\right] \mathrm{d}t \tag{15}$$
$$\text{s.t.} \quad \dot{s} = 4s(t)(1 - c(t)) \tag{16}$$
$$s(0) = 1 \tag{17}$$
$$s(1) = e^2. \tag{18}$$

Hints:

- At some point your necessary conditions will lead to a system of two differential equations.
- The equation for the *co-state* can be simplified to be independent of the *state* and can then be solved on its own.
- This solution can be substituted into the second differential equation. The latter equation can be solved explicitly.
- The final numerical solutions for the constants are *not* smooth / round.