Exam: Methods for Economists (20312)
Examiner: PD Dr. Jäger-Ambrożewicz **Summerterm 2012**

Remarks:

1. The exam comprises 6 problems on 2 pages. You have to solve all of them.
2. Allowed aids: bilingual dictionary and pocket calculator according to the rules of the examination office.
3. Your time budget is 120 minutes.
4. When answering the questions below it is necessary that explanations and intermediate steps are provided!
5. Good luck!

Problem 1: (20 min)
Consider the following system of equations:

$$xu + 2y - uv + v^3 = 1 \qquad (1)$$
$$x^3 + y^3 + uv = 2. \qquad (2)$$

a) Use the implicit function theorem to show that the system defines u and v as functions of x and y in a neigbourhood around $(x, y, u, v) = (1, 0, 1, 1)$.

b) Find the values of the partial derivatives of the two functions u and v w.r.t. x and y when $x = 1$, $y = 0$, $u = 1$, $v = 1$.

Problem 2: (15 min)
Consider the following small macroeconomic IS-LM model:

$$Y = C(Y) + I(r) + G \qquad \text{(IS curve (IS))}$$
$$\frac{M}{P} = L(r, Y) \qquad \text{(LM curve (LM))},$$

where $0 < C'(Y) < 1$, $I'(r) < 0$, $L_r < 0$, $L_Y > 0$.

a) Calculate the slope $\left.\frac{\mathrm{d}r}{\mathrm{d}Y}\right|_{I=S}$ of the IS-curve (IS).

b) Calculate the slope $\left.\frac{\mathrm{d}r}{\mathrm{d}Y}\right|_{\frac{M}{P}=L}$ of the LM-curve (LM).

c) Use **the Implicit Function Theorem** to calculate the derivative $\frac{\mathrm{d}Y}{\mathrm{d}G}$.

Problem 3: (20 min)
Consider the following differential equation:

$$\dot{x} = x \cdot t + t. \qquad (3)$$

a) Draw a slope diagramm.

b) Is the differential equation seperable? Base your answer on verbal or analytic reasoning.

c) Find the solution to the initial value problem with $x(0) = 1$.

Problem 4: (20 min)
Let D denote the debt ratio (i.e. debt/income), $d \in \mathbb{R}$ denote the deficit ratio (deficit/income), r denote the interest rate and g denote the growth rate of income. Suppose that d, r, g are constant. The following differential equation describes the dynamics of the debt ratio.

$$\dot{D} = d + (r - g)D. \tag{4}$$

a) Consider the case where $g > r$. Draw a phase diagram and argue why debt is no problem.

b) Consider the case where $r > g$ and $d > 0$. Draw a phase diagram and argue why debt is a problem.

c) Still consider the case where $r > g$ and suppose that $d < 0$ (a primary surplus). Draw a phase diagram and argue that debt is a problem if D is beyond a specific limit. Calculate that limit.

Problem 5: (25 min)

a) Consider the following maximization problem:

$$\max_{x_1, x_2} \quad \pi = x_1 \tag{5}$$
$$\text{s. t.} \quad x_1^2 + x_2^2 \leq 1 \tag{6}$$
$$\text{and} \quad x_1, x_2 \geq 0. \tag{7}$$

a 1) Find the optimal solution using *First Order Conditions / Kuhn-Tucker-Conditions* and graphical analysis (also explaining the graph of course).

a 2) Check whether the solution found in 1a satisfies the constraint qualification.

b) Consider the following minimization problem:

$$\min_{x_1, x_2} \quad C = x_1 \tag{8}$$
$$\text{s. t.} \quad x_1^2 - x_2 \geq 0 \tag{9}$$
$$\text{and} \quad x_1, x_2 \geq 0. \tag{10}$$

b 1) Solve the problem graphically (also explaining the graph of course).

b 2) The optimal solution is different in character when compared to the solution found in 1a. In what way is the solution different in character and how could one deal with this issue?

Problem 6: (25 min)
Consider the following optimal control problem, where u denotes the control and x denotes the state:

$$\max_{u} \quad \int_0^1 (1 - tx - u(t)^2)\, dt \tag{11}$$
$$\text{s. t.} \quad \dot{x} = u(t), \tag{12}$$
$$x(0) = 0 \tag{13}$$
$$\text{and} \quad x(1) = \text{free}. \tag{14}$$

Hints: Note the terminal condition. With this terminal condition it follows $p(1) = 0$, where p denotes the costate.

a) Solve the optimal control problem

b) Consider the hint and explain the economic intuition of $p(1) = 0$.